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REMARKS ON A PROOF THAT A CONTINUOUS FUNCTION IS UNIFORMLY CONTINUOUS

By N. J. Lennes

1. In a note in the *Mathematische Annalen*, vol. 6 (1873), p. 319, Lüroth gave a proof* of the theorem that a function, continuous over a certain interval, is uniformly continuous over that interval. The proof is based upon the following definition of continuity at a point:

A single valued function,† $f(x)$, *is said to be continuous at a given point* $x = x_0$ *if for every positive* ϵ *there exists a positive* δ_ϵ *such that for every* x_1 *and* x_2 *on the interval* $x_0 - \delta_\epsilon \cdots x_0 + \delta_\epsilon$‡ *we have*

$$| f(x_1) - f(x_2) | < \epsilon.$$

In other words, the range§ of the function on the interval $x_0 - \delta_\epsilon \cdots x_0 + \delta_\epsilon$ is less than ϵ.

If one value of δ_ϵ can be found which satisfies the condition for the given point x_0, then clearly every smaller value, and possibly some larger ones, will satisfy the condition. Let Δ_ϵ be the largest value of δ_ϵ which satisfies the condition.‖

If the given function $f(x)$ is continuous at every point on $a \cdots b$, then for any ϵ there will be a value of Δ_ϵ corresponding to every x on $a \cdots b$. This Δ_ϵ then for any particular ϵ will be a function of x and may be denoted by $\Delta_\epsilon(x)$.

The essential part of Lüroth's proof consists in establishing the following fact: if $f(x)$ is continuous at every point of its interval, then for any particular value of ϵ, the quantity $\Delta_\epsilon(x)$ will be a continuous function of x. From this follows, by a fundamental theorem due to Weierstrass, that the function $\Delta_\epsilon(x)$ will actually reach its greatest lower bound, that is, will have a minimum value; and this minimum value, like all the other values of δ_ϵ, will be a positive quantity.

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*Lüroth's proof is stated for functions of two variables, but applies, as he himself points out, to functions of any number of variables.

†The function $f(x)$ is supposed to be defined for every point of a given continuous interval $a \ldots b$. (We confine ourselves wholly to functions of one real variable.)

‡Or on such parts of $x_0 - \delta_\epsilon \ldots x_0 + \delta_\epsilon$ as lie within $a \ldots b$.

§ The expression "range" of a function on a given interval is used to indicate the difference between the least upper and the greatest lower bound of the function on that interval.

‖For particular values of ϵ and x_0, Δ_ϵ may clearly equal the whole distance from x_0 to "a" or "b" This is so for all values of ϵ and x_0, when $f(x)$ is a constant.

Hence: *if a single valued function is continuous over an interval* $a \cdots b$, *then for every* ϵ *there exists a positive* η [*namely, twice the minimum value of* $\Delta_\epsilon(x)$] *such that the range of the function will be less than* ϵ *on any interval on* $a \cdots b$ *whose length is less than* η. That is, the function is uniformly continuous and the theorem is proved.

2. A definition of continuity more familiar than that used by Lüroth is the following: *A single valued function* $f(x)$ *is said to be continuous at a point* $x = x_0$ *when for every positive* ϵ *there exists a positive* δ'_ϵ *such that for every* x_1 *on the interval* $x_0 - \delta'_\epsilon \cdots x_0 + \delta'_\epsilon$ *we have*

$$| f(x_0) - (fx_1) | < \epsilon ;$$

and the question arises whether this definition might have been used in place of the other in Lüroth's proof of the theorem. We may here, as before, consider the greatest δ'_ϵ which satisfies the condition, and denote it by $\Delta'_\epsilon(x)$. The question then is whether this quantity $\Delta'_\epsilon(x)$, like $\Delta_\epsilon(x)$, is a continuous function of x.

The purpose of the present note is to show that this question must be answered in the negative. The following examples prove that $\Delta'_\epsilon(x)$ is not in general a continuous function of x.

Consider the function $f(x) = \sin x$, from $x = 0$ to $x = \pi$, and take $\epsilon = \frac{1}{2}\sqrt{3}$. At $x = \frac{\pi}{3}$, $\Delta'_\epsilon(x) = \frac{2\pi}{3}$, while if x is allowed to approach $\frac{\pi}{3}$ as a limit $\left(x \text{ being greater than } \frac{\pi}{3}\right)$, $\Delta'_\epsilon(x)$ approaches $\frac{\pi}{3}$ as a limit. Hence $\Delta'_\epsilon(x)$ is discontinuous at $x = \frac{\pi}{3}$.

As another example, consider a function $f(x)$ which is represented by the following broken line. On a segment AB parallel to the X-axis there are two points C and D. On the segment CD as a base construct an isosceles triangle CED. The segments AC, CE, ED and DB, including their end points, represent a function $y = f(x)$ such that for certain values of ϵ, $\Delta'_\epsilon(x)$ is not a continuous function on the interval $A \cdots B$. This case is strictly analogous to the one given above.

Instead of the one triangle DEC we now construct an infinite set of triangles as follows:

On the segment AB lay off segments BX_1, $X_1X_2 \cdots, X_nX_{n+1}, \cdots$ such $X_nX_{n+1} = \dfrac{X_{n-1}X_n}{2}$. On segment BX_1, H_1 is any point. K_n is a point on X_nX_{n+1} such that $X_nK_n = \dfrac{X_nX_{n+1}}{3}$; $H_n(n \geqq 2)$ is a point on $X_{n-1}\,X_n$ such that $H_n\,X_n = \dfrac{X_{n-1}X_n}{3}$.

On the H_nK_n as bases construct isosceles triangles with vertices V_n and altitudes h_n such that $h_n = \frac{2}{3}(h_{n-1})$.

The segments BH_1, H_1V_1, V_1K_1, $K_1H_2, \cdots, H_nV_n$, V_nK_n, K_nH_{n+1}, $\cdots$ including their end points represent a function $f(x)$ such that for *every* value of ϵ which is less than the range of the function on its interval, there is a point at which $\Delta'_\epsilon(x)$ is a discontinuous function of x.

The function $y = x \sin \dfrac{1}{x}$ is such that for some values of ϵ within an interval $0 \cdots \Delta$ [Δ being positive and not zero], $\Delta_\epsilon(x)$ is discontinuous at some point no matter how small Δ is. This is also true for ϵ on an interval $\Delta \cdots +\infty$ no matter how large Δ is.

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